Generation of entangled states and error protection from adiabatic avoided level crossings

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We consider the environment-affected dynamics of N self-interacting particles living in one-dimensional double wells. Two topics are dealt with. First, we consider the production of entangled states of two-level systems. We show that by adiabatically varying the well biases we may dynamically generate maximally entangled states, starting from initially unentangled product states. Entanglement degradation due to a common type of environmental influence is then computed by solving a master equation. However, we also demonstrate that entanglement production is unaffected if the system-environment coupling is of the type that induces "motional narrowing". As our second but related topic, we construct a different master equation that seamlessly merges error protection/detection dynamics for quantum information with the environmental couplings responsible for producing the errors in the first place. Adiabatic avoided crossing schemes are used in both topics.

I. INTRODUCTION

Real quantum systems are always coupled to environments. Research in the last two or three decades has shown that system–environment interactions greatly affect quantal coherences in the system. Open-system dynamics can often be described by a master equation for the reduced density matrix of the system. The full Hamiltonian is written as the sum of system-only, environment-only and system–environment interaction terms. The master equation for the system then depends on the system-only Hamiltonian plus other terms derived from the fundamental system–environment couplings. These additional pieces induce an effective non-unitary evolution for the system that can, for example, lead to an apparent loss of quantal coherence.

In many studies, the system-only Hamiltonian is taken to be the sum of one-body terms. An example of relevance to this paper arises for a system of N particles, each in a one-dimensional symmetric double well. The system Hamiltonian

$$H_{\rm sys} = \sum_{i=1}^{N} \left(E^{(i)}\, \mathbb{1}^{(i)} + \omega^{(i)} \sigma_x^{(i)} \right) \tag{1}$$

describes particle i oscillating with frequency ω_i between the left and right sides of its well independently of the other particles in the system. When system–environment coupling is added, this behaviour can be drastically modified, with the quantum Zeno or freezing phenomenon being the most extreme example.

The purpose of this paper is to contribute to a study of closed and open quantum system evolution that goes beyond one-body system Hamiltonians by including interaction terms between the system particles. This is a vast and extremely rich field of enquiry, with many possible lines of development. For reasons to be explained below we will focus on two topics. The first is entanglement creation through adiabatic evolution and its degradation due to environmental influences. The second is quantum information error protection through adiabatic evolution.

Common to both topics is the use of adiabatic avoided level crossing dynamics. This special case of quantum mechanical time evolution has applications in diverse areas of physics, from the Mikheyev-Smirnov-Wolfenstein effect for neutrino oscillations [1] to quantum computation [2]. We use it here because it allows non-trivial dynamics to occur while retaining both an element of theoretical simplicity plus visualisability with the aid of level crossing diagrams.

Whenever one has more than a single particle in one's quantum system, entanglement adds spice to the analysis. Entangled states are a unique feature of quantum mechanics, first studied in connection with the issue of non-locality [3]. They are also of critical importance in discussions of decoherence, measurement and the quantum-to-classical transition [4]. More recently, entanglement has emerged as a useful resource in the field of quantum information theory, enabling processes such as quantum cryptography [5] and teleportation [6]. The production and manipulation of entangled states is a key element in any realisation of a quantum computer [7].

As our first topic, we describe an efficient method for the preparation of entangled states which utilises the level crossing structure of the system's energy eigenstates. Specifically, we discuss the Bell states in the case of a bipartite system, W states in a tripartite system and generalised W_N states in multi-partite systems. We then subject the evolving systems to environmental interactions that affect the entanglement production process. We show that the outcome depends on the form of system-environment coupling: entanglement degradation is the generic effect, but the special case of "motional narrowing" noise actually leads to zero degradation. Calculations presented in Ref. [9] simply took initial Bell states and computed the evolution of entanglement in the presence of environmental noise. The present work is a natural extension of this.

As a separate but related topic, we show that a method of error protection using entangled states emerges from a variation of the above procedure. We devise a scheme to protect the information encoded in a single qubit from bitflip and phaseflip errors. Our goal is to construct a master equation which seamlessly joins the dynamics of qubit encoding, error protection and decoding to the environment-coupling influences responsible for error generation. While the outcome of the open system evolution can be dissected into these quantum information theoretic subprocesses, the actual dynamics is simply what you get when you solve a certain master equation.

The building block for our multi-partite systems is the two-state system of a particle in a one-dimensional double well, chosen for its generic properties and broad interest (see Ref.

[8]). The well is taken to be partitioned by a barrier of height V that is large compared to the ground state energy E_0. (The two lowest energy levels will be collectively called the "ground state". For a symmetric well with $V \gg E_0$ they are nearly degenerate.) Adopting quantum information theoretic parlance, we denote by "qubits" the states occupying the two-dimensional Hilbert space spanned by the left and right sides of the well (in the ground state). The left and right eigenstates, denoted $|1\rangle$ and $|0\rangle$, correspond to the $+1$ and -1 eigenvalues of σ_z, respectively. For an isolated well, the Hamiltonian has the form

$$H_1 = E_0 \, \mathbb{1} + \omega\sigma_x + f\sigma_z, \tag{2}$$

where the energy splitting ω is essentially a tunneling rate between the two sides of the well, and the parameter f is the well bias.

II. GENERATION OF ENTANGLED STATES

A. Two qubit system

We begin with a bipartite two-state system, the simplest system in which one may discuss entanglement. Our aim here is to produce one of the Bell states,

$$|\Psi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|10\rangle \pm |01\rangle), \tag{3}$$

dynamically from certain initially factorisable states.

Consider a Hamiltonian of the form presented in Eq. (2) for each of the two qubits, together with a third term which couples them:

$$H_2 = [\omega\sigma_{x1} + f(t)\sigma_{z1}] \otimes \mathbb{1} + \mathbb{1} \otimes [\omega\sigma_{x2} + f(t)\sigma_{z2}] + \lambda\sigma_{z1} \otimes \sigma_{z2}. \tag{4}$$

Observe that the Hamiltonian is invariant under the interchange of the two qubits. For simplicity, only the well biases f are allowed to vary with time, and we demand that $\omega \lesssim 0.1\lambda$. This last condition is necessary in order for the desired level crossing structure to arise. (It is implicitly assumed that only the lowest states in each well are populated. We will need to relax this condition later on when we couple the system to a hot bath.)

The four eigenstates separate into an antisymmetric singlet state $|\Psi^-\rangle$ for all values of ω, f and λ, and a symmetric triplet composed of linear superpositions of the set $\mathcal{S} = \{|11\rangle, |00\rangle, |\Psi^+\rangle\}$, which asymptotes to $\mathcal{S}$ for large $|f|$'s. Note that the permutation symmetry of the Hamiltonian ensures that the evolution of the singlet state is disconnected from that of the triplet states at all times so we are dealing with an effectively three-state system. In Fig. 1, we plot the energy level diagram as a function of f.

The states $|11\rangle$ and $|00\rangle$ experience avoided level crossings with $|\Psi^+\rangle$ at $f = -\lambda$ and $f = \lambda$ respectively, labelled points A and C in Fig. 1. Hence, an initial state $|\phi(t=0)\rangle = |11\rangle$ created at $f \ll -\lambda$ will continue to reside in the lowest energy eigenstate provided that f varies sufficiently slowly with time, and be adiabatically transformed, across point A, to the entangled state

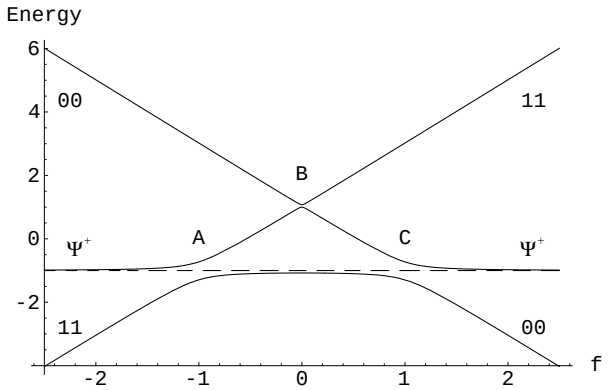

FIG. 1. Level crossing diagram for the two-qubit system. The variable f is in units of λ, and the dotted line denotes the decoupled state $|\Psi^-\rangle$.

$$|\phi(t=T)\rangle \simeq |\Psi^+\rangle - \frac{\omega}{\sqrt{2}(f+\lambda)}|11\rangle + \frac{\omega}{\sqrt{2}(f-\lambda)}|00\rangle + \mathcal{O}(\omega^2), \tag{5}$$

where T corresponds to any point in the period of time during which f lies in the range $-\lambda < f < \lambda$. Further adiabatic evolution beyond point C will turn $|\phi(t)\rangle$ once more into a factorisable state $|00\rangle$. Note in Eq. (5) that the entangled state $|\phi(t=T)\rangle$ inevitably contains some order $\omega/[f+\lambda, f-\lambda]$ "contamination" from the states $|11\rangle$ and $|00\rangle$, which cannot be arbitrarily minimised without jeopardising the adiabaticity of the transitions at points A and C (see later).

Figure 2 illustrates the evolution of the entanglement for $|\phi(t)\rangle$, and we remind the reader again that the Bell state $|\Psi^-\rangle$ cannot be generated in this manner because of symmetry requirements. In this figure, the degree of entanglement is defined to be the *entropy of entanglement* E which is the von Neumann entropy of the reduced density matrix obtained by tracing out either of the two qubits [10],

$$E = -\mathrm{Tr}\rho_A \log_2 \rho_A = -\mathrm{Tr}\rho_B \log_2 \rho_B. \tag{6}$$

The related concept, *entanglement of formation* E_f, to be used below, is a generalisation of the entropy of entanglement for a *mixed* state of two qubits [11].

The entanglement will stay approximately constant at $E \simeq 1$ when f is in the range $-\lambda < f < \lambda$. Note that if the two qubits had different oscillations frequences, $\omega^{(i)}$, mixing between the almost degenerate states $|\Psi^\pm\rangle$ is not prohibited by permutation symmetry (or the lack thereof). In this case, the degree of entanglement will vary significantly in the period $-\lambda < f < \lambda$, attaining the maximal value of $E \simeq 1$ only at the point $f = 0$.

Observe that if we had started instead in the state $|00\rangle$ which is approximately the highest energy eigenstate, we could in principle obtain the entangled state $\frac{1}{\sqrt{2}}(|11\rangle + |00\rangle)$ at the point $f = 0$. However, an inspection of the Hamiltonian

$$H_2^{\mathrm{sym}} = \left(\begin{array}{ccc|c} 2f+\lambda & \sqrt{2}\omega & 0 & \\ \sqrt{2}\omega & -\lambda & \sqrt{2}\omega & 0 \\ 0 & \sqrt{2}\omega & -2f+\lambda & \\ \hline & 0 & & -\lambda \end{array}\right), \tag{7}$$

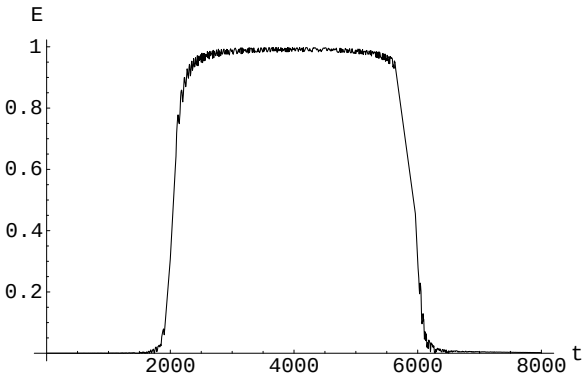

FIG. 2. Entanglement of $|\phi(t)\rangle$, where $|\phi(0)\rangle = |11\rangle$, as a function of time. We take the parameters $\omega = 0.05$, $\lambda = 1$, and $f(t) = -2 + t/2000$, where f, ω and λ are in units of an arbitrary energy normalisation $E_{\rm norm}$ and the time t is in units of $1/E_{\rm norm}$.

rewritten in the symmetrised basis $\{|11\rangle, |\Psi^+\rangle, |00\rangle, |\Psi^-\rangle\}$, shows that the splitting between the relevant energy eigenstates for the $|00\rangle \rightleftharpoons |11\rangle$ transition at $f = 0$ is of order ω^2/λ, as opposed to $\sqrt{2}\omega$ for the $|11\rangle \rightleftharpoons |\Psi^+\rangle$ transition considered earlier. It follows that the resonance width at point B is necessarily some ω/λ times narrower than those at points A and C. Their respective adiabaticity parameters, defined as

$$\gamma \equiv \left| \frac{(\Delta k)^2}{\frac{d}{dt}\sqrt{(\Delta k)^2 - (\Delta k|_{\rm res})^2}} \right|_{\rm res}, \tag{8}$$

where Δk is the splitting between the energy eigenstates, "res" denotes evaluated at resonance, and $\gamma > 1$ indicates an adiabatic process, must also differ correspondingly by some factor of ω^2/λ^2. Explicitly,

$$\gamma_{A,C} \sim \frac{4\omega^2}{|\dot{f}|}, \qquad \gamma_B \sim \frac{2\omega^4}{\lambda^2}\frac{1}{|\dot{f}|} \tag{9}$$

characterise the three resonances.

Hence, in order to manufacture the state $\frac{1}{\sqrt{2}}(|11\rangle + |00\rangle)$, a slowly varying f together with control over its accuracy to within $\pm\omega^2/\lambda$ are necessary. Naïvely, the severity of the precision requirement seems easily alleviated by enlarging ω. The benefit of a wider resonance width, however, is compensated for by a concomitant increase in the contamination from the $|\Psi^+\rangle$ state which contributes at order ω/λ. By comparison, the production of $|\Psi^+\rangle$ from $|11\rangle$ requires only that we stop the evolution somewhere between $-\lambda < f < \lambda$, and is therefore a much simpler task.

1. System-Environment coupling: cold bath

In the presence of environmentally-induced noise, we must solve a master equation for the reduced density matrix of the system.[1] The form of the master equation depends on whether or not the environment is hot enough to induce transitions to excited states of the double wells. We first consider coupling to a cold environment, so that each particle remains in the ground state of its double well.

Denote the two-particle (reduced) density matrix of qubits A and B as

$$\rho = \rho_A \otimes \rho_B. \tag{10}$$

The evolution of ρ is governed by

$$\frac{d}{dt}\rho(t) = -i[H, \rho(t)] - \Gamma_{\rm relax}[\zeta, [\zeta, \rho]] - \Gamma_{\rm relax}[\zeta', [\zeta', \rho]], \tag{11}$$

where ζ and ζ' are Hermitian matrices obtained from the system-environment interaction Hamiltonian. For the sake of the example, we will choose $\zeta = \frac{1}{2}(\mathbb{1} + \sigma_3) \otimes \mathbb{1}$, $\zeta' = \mathbb{1} \otimes \frac{1}{2}(\mathbb{1} + \sigma_3)$, which means that the relaxation terms arise from a system–bath interaction that couples to the left states $|1\rangle$ but not to the right states $|0\rangle$.[2] Commencing with a pure state $|11\rangle$ and a small relaxation rate $\Gamma_{\rm relax}$, we show in Fig. 3 the evolution of the entanglement with the course of time. A "decay" in E in the period $-\lambda < f < \lambda$ is evident. For a sufficiently large $\Gamma_{\rm relax}$, the density matrix simply becomes proportional to the unit matrix, and the entanglement goes to zero at all times.

2. System-Environment coupling: hot bath

We now consider coupling to an environment hot enough that the system-bath interactions will cause particles in the wells to populate states higher than the ground state. In a previous publication we considered in detail the effects of noise coming from a thermal bath coupled to the full excited state spectrum of a double well [9]. Among the cases considered was the effect of "side-blind" noise on a number of properties of the system. For the side-blind case, the bath couples *equally* to the left states $|1\rangle$ and the right states $|0\rangle$.[3] If we begin with a single particle on the left in the ground state of the symmetrical well, the thermal excitation of states that have very different R-L oscillation frequencies replaces with a

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[1] Certain conditions must be satisfied for the dynamics of an open quantum system to be well-described by a master equation. We consider only such cases in this paper.

[2] The other extreme has the environment coupling equally strongly to the left and right sides of the wells, so that ζ and ζ' are proportional to the identity, and the double-commutator terms vanish. We will return to this point shortly.

[3] In the absence of excitation to higher energy states, side-blind noise does nothing of note.

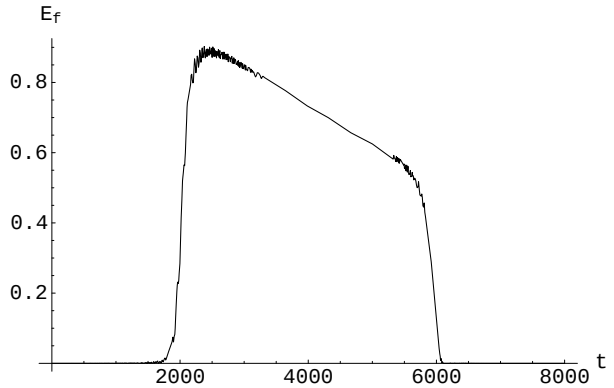

FIG. 3. Entanglement of formation as a function of time in the presence of noise. We take $\Gamma_{\rm relax} = 0.5 \times 10^{-4} E_{\rm norm}$, with the other parameters the same as in Fig. 2

multiperiodic tangle the previous simple oscillation of the probability of finding the particle on the left at a later time. The effects of weak coupling to the noise gradually damp these oscillations, leaving a time independent 50% occupancy on either side. As the strength of the noise-system coupling is increased, however, the motion becomes more organized, and in the limiting case of very large noise, we retrieve sinusoidal oscillation with a frequency that is given by the thermal average of the frequencies of the individual levels [9]. The analogous phenomenon is called "motional narrowing" in NMR studies.

We will now look at entanglement-formation dynamics in a two-particle system with excited states. The L-R oscillation properties for a particular well-level are given by Eq. (11), but the oscillation parameters are different for the excited well states. We shall refer to the variable distinguishing these states as the "vertical" variable, as opposed to the (L, R) degree of freedom which we will term the "horizontal" variable. We will take the coupling among the vertical levels to again be "side-blind" , that is, not dependent on the horizontal variables, because that is the most interesting case. We can capture the essential features of the problem by considering only two vertical levels, with equal transition rates, Γ, each to the other [the latter as in the case of a high temperature for the thermal bath), in which case we make the replacements in Eq. (11] , $\rho(t) \rightarrow \rho(E_i, t)$, $i = 1, 2$ on the lefthand side and

$$\Gamma_{\rm relax}[\zeta, [\zeta, \rho]] \rightarrow \Gamma\Big(\sum_j \rho(E_j, t) - 2\rho(E_i, t)\Big) \tag{12}$$

on the righthand side, as well as specifying, within H, different oscillation parameters for the two energy states.

Using this equation we now look at the results for the two particle-entanglement for the cases of zero, moderate, and strong bath coupling strengths. The results are shown in Fig. 4. We see substantial degradation of entanglement-formation in the case of moderate coupling. But in the case of strong coupling, we see a pattern of entanglement creation that is similar to that in the case with no noise. In this limit, the system is made to change vertical state so rapidly that it can only respond to an average Hamiltonian, with quantal coherence fully maintained [9]. For these reasons, the entanglement-formation dynamics succeeds in this

limit.

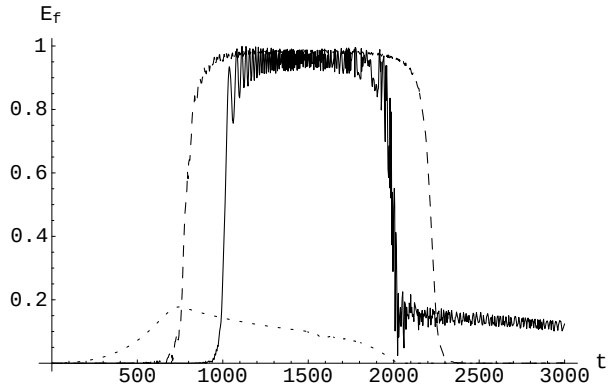

FIG. 4. Entanglement of formation, E_f as a function of time, in the presence of "side-blind" noise. The solid, dotted and dashed curved are for zero, moderate amd large system-bath interaction rates, $\Gamma = 0$, 1 and 1000 respectively. For simplicity, we took a system with only two vertical states, and the initial condition was chosen to be $|11\rangle$, with both particles in their ground states. We have taken the parameters $\omega_0 \equiv \omega(E_0) = 0.05$, $\omega_1 \equiv \omega(E_1) = 0.1$, $\lambda(E_0, E_1) = 20(\omega_0\omega_1)^{0.5}$ and $f = -3 + t/500$, where ω, f, λ and Γ are in in units of an arbitrary energy parameter $E_{\rm norm}$, and the time in units of $1/E_{\rm norm}$. Note that the position of the level crossing differs for the solid and dashed curves as λ has been taken to be dependent on the value of the vertical variable.

For completeness, we mention that if the bath does not couple with equal strength to the two horizontal states $|0\rangle$ and $|1\rangle$, the behaviour is qualitatively the same as that obtained from Eq. (11) for the cold bath, where $\zeta \neq \mathbb{1}$.

B. Multi-Qubit system

Consider now the W states,

$$|W\rangle_{001} = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle),$$
$$|W\rangle_{110} = \frac{1}{\sqrt{3}}(|110\rangle + |101\rangle + |011\rangle). \qquad (13)$$

These may be considered maximally entangled 3-qubit states, in the sense that each has the maximum amount of bipartite entanglement if any one of the three qubits is traced out [12].

It turns out that the production of a three-qubit W state from an initially factorisable state such as $|111\rangle$ via adiabatic transitions is completely analogous to the generation of the symmetric Bell state from $|11\rangle$ considered previously. We now elaborate on this point.

The Hamiltonian for our three-qubit system is a direct generalisation of that in Eq. (4):

$$H_3 = \omega\left[\sigma_{x1} + \sigma_{x2} + \sigma_{x3}\right] + f(t)\left[\sigma_{z1} + \sigma_{z1} + \sigma_{z3}\right] + \lambda\left[\sigma_{z1}\sigma_{z2} + \sigma_{z2}\sigma_{z3} + \sigma_{z3}\sigma_{z1}\right], \qquad (14)$$

where, for simplicity, we are dealing explicitly with the permutation invariant case. Decomposition of the tensor product $2 \otimes 2 \otimes 2 = 4 \oplus 2 \oplus 2$ reveals that the permutation symmetric

set $\{|111\rangle, |W\rangle_{110}, |W\rangle_{001}, |000\rangle\}$ can be treated in isolation, with its evolution governed by the 4×4 block in the Hamiltonian

$$H_3^{\rm sym} = \begin{pmatrix} 3f+3\lambda & \sqrt{3}\omega & 0 & 0 \\ \sqrt{3}\omega & f-\lambda & 2\omega & 0 \\ 0 & 2\omega & -f-\lambda & \sqrt{3}\omega \\ 0 & 0 & \sqrt{3}\omega & -3f+3\lambda \end{pmatrix}, \tag{15}$$

written in the symmetrised basis. The level crossing diagram for this set of states is plotted in Fig.5.

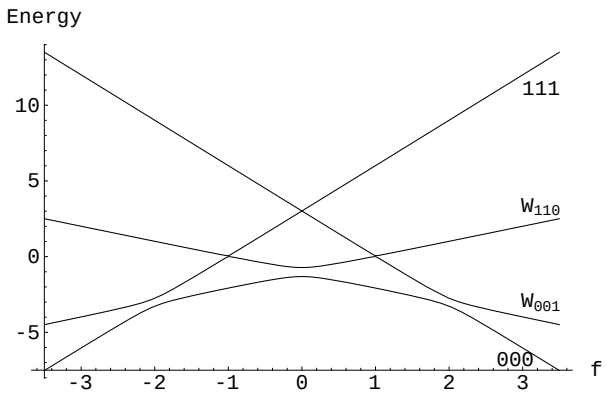

FIG. 5. Energy level diagram for the set of four permutation symmetric states of the three-qubit system, as a function of the well bias f. Note that none of the levels cross.

Three resonances mark the evolution of the lowest energy eigenstate from $f < -2\lambda$ to $f > 2\lambda$. To zeroeth order in ω, the identity of the state is adiabatically transformed as per the evolutionary sequence

$$|111\rangle \stackrel{f=-2\lambda}{\longrightarrow} |W\rangle_{110} \stackrel{f=0}{\longrightarrow} |W\rangle_{001} \stackrel{f=2\lambda}{\longrightarrow} |000\rangle, \tag{16}$$

where the conditions above the arrows denote the points at which the transitions take place. Commencing with the product state $|111\rangle$ at $f < -2\lambda$, the two W states can be easily obtained respectively in the domains $-2\lambda < f < 0$ and $0 < f < 2\lambda$.

Note in passing that the state $|{\rm GHZ}\rangle = |111\rangle + |000\rangle$ [13] coincides approximately with an eigenstate at $f = 0$ and can in principle be accessed from $|000\rangle$ or $|111\rangle$ via the highest energy eigenstate. However, as a direct analogue of the $\frac{1}{\sqrt{2}}(|11\rangle + |00\rangle)$ state considered in the previous section, the production of the GHZ state by our method suffers the same problem of non-adiabaticity and again requires precise control over the value of f.

The generalised form of the W state for N qubits is defined as

$$|W\rangle_N = \frac{1}{\sqrt{N}}|N-1, 1\rangle, \tag{17}$$

where $|N-1, 1\rangle$ denotes the totally symmetric state with $N-1$ qubits in the state $|0\rangle$ and one qubit in the state $|1\rangle$.

The residual bipartite entanglement between any two pairs of qubits, upon tracing the other qubits out, has been shown to have the maximum value attainable in an N qubit system [14]. Since $|W\rangle_N$ states are symmetric with respect to permutation of the N qubits, they can always be made from $|11\cdots1\rangle$ or $|00\cdots0\rangle$ in the manner discussed above.

III. ERROR DETECTION AND CORRECTION

An important problem in the study of quantum information theory is the protection of information from various types of errors. Any practical realisation of a quantum computer will see the physical systems, the states of which comprise the qubits, experience effective violation of unitary evolution and error generation, both due to coupling to their environment. A number of error detection and protection schemes have been devised that encode a given qubit through entanglement with other otherwise redundant qubits [4] [15]. Our goal in this section is to construct a master equation that seamlessly merges error generation, protection through entanglement and detection within a single dynamical framework. While the resulting evolution can be usefully understood through reduction to information theoretic subprocesses, at the level of pure physics the single equation that one solves encapsulates both the quantum processes over which the experimenter has no control, and the knobs the experimentalist turns to change the biases.

In what follows, averted level crossing schemes similar to those described in the previous section are used to encode a qubit in a way that it is protected from certain outside disturbances, following the steps:

1. We consider an initial pure state given by $|\Psi\rangle = a|11\rangle + b|01\rangle \equiv [a|1\rangle + b|0\rangle]|1\rangle$. Our goal is to preserve the information encoded in the first qubit.

2. We take a noise term that is derived from an external bath that flips qubits, independently, for one or both particles, and we follow the density matrix for the resulting mixed state.

3. We transform the state in the time interval $t = 0$ to $t = t_e$ by adiabatically changing the coefficients of certain terms in the Hamiltonian; we refer to these functions as the bias functions. Next we leave the system in this state for a length of time t_h ("hang time") *long* compared to t_e.

4. To reclaim the initial state we adiabatically restore the biases to their initial values and then measure the value of the second, control, coordinate. If the measurement gives back the original state $|1\rangle$, then for the first qubit we obtain the state that is the projection of our evolved density matrix on that state. The coding algorithm is such that this operation reclaims the density matrix of the initial pure state, with small degradations coming from the small non-adiabaticity due to doing the encoding and decoding over finite time intervals, or with degradations from letting the hang time be so large that multiple errors are likely to have accumulated. If, on the other hand, we measure the value $|0\rangle$ for the control state, then, beginning from the projection of our evolved density matrix on the state $|0\rangle$ a simple, and implementable unitary operation restores the initial density matrix.

We begin this process by describing the system in terms of the four time dependent instantaneous eigenstates of the Hamiltonian that describes the system without noise, $|\xi_j\rangle_t$, where $j = 1, 4$. Initially, we take $|\xi_1\rangle_0 \equiv |\xi_1\rangle_{t=0} = |1\rangle \otimes |1\rangle$, $|\xi_2\rangle_0 = |0\rangle \otimes |1\rangle$, $|\xi_3\rangle_0 = |1\rangle \otimes |0\rangle$, $|\xi_4\rangle_0 = |0\rangle \otimes |0\rangle$, where, as indicated above, the initial data is to be encoded in a combination of $|\xi_1\rangle_0$, and $|\xi_2\rangle_0$. It is useful in what follows to define a time dependent projection operator on the subspace used for encoding, $P_e(t) \equiv |\xi_1\rangle_t\langle\xi_1|_t + |\xi_2\rangle_t\langle\xi_2|_t$. We note that $P_0 \equiv P_e(0) = \mathbb{1} \otimes |1\rangle\langle 1|$, that is to say, the projection operator on the value unity for the second qubit.

In our example, we will construct a dynamics that protects against the "bit-flip" errors defined by the error operators $\sigma_x \otimes \mathbb{1}$ and $\mathbb{1} \otimes \sigma_x$, plus "phase errors" induced by $\mathbb{1} \otimes \sigma_z$ and $\sigma_z \otimes \mathbb{1}$. (It is easy to modify the scheme to protect against all four bit-flip errors $\sigma_{x,y} \otimes \mathbb{1}$ and $\mathbb{1} \otimes \sigma_{x,y}$, but at the cost of losing the phase error protection.) To protect against one of these errors we need to have all matrix elements of the error operator in question equal to zero in the subspace spanned by $|\xi_1\rangle_{t_e}$ and $|\xi_2\rangle_{t_e}$. There will of course be nonvanishing matrix elements from this subspace to the subspace spanned by $|\xi_3\rangle_{t_e}$ and $|\xi_4\rangle_{t_e}$. We can obtain these properties by choosing the Hamiltonian

$$H = f(t)\, g_{11}\, \sigma_x \otimes \sigma_x + g_{33}\, \sigma_z \otimes \sigma_z + [1 - f(t)]\, h\, \sigma_z \otimes \mathbb{1}\,, \tag{18}$$

where $f(t)$ is a function that begins at zero at $t = 0$, rises to unity at $t = t_e$, remains unity until $t = t_e + t_h$, and then decreases back to zero at $t = 2t_e + t_h$. If we pick, for example $g_{11} = 2$, $g_{33} = 1$ and $h = 4$, then the eigenstates that at $t = 0$ were $|\xi_1\rangle_0$ and $|\xi_2\rangle_0$ become, respectively, $(|11\rangle + |00\rangle)/\sqrt{2}$ and $(|10\rangle - |01\rangle)/\sqrt{2}$ at time $t = t_e$, modulo all-over phase factors, and remain so until $t = t_e + t_h$. We note that all four of the chosen error operators have vanishing matrix elements in this subspace of states.

We now give the analytical detail behind the above assertions, and provide the basis for some quantitative examples given below. For clarity of exposition we shall leave the noise turned off during the short encoding-decoding times; however in the numerical example presented later, we retain the noise throughout the process. We begin with a density matrix,

$$\rho(0) = [a|\xi_1\rangle_0 + b|\xi_2\rangle_0]\,[a^*\langle\xi_1|_0 + b^*\langle\xi_2|_0]\,, \tag{19}$$

and let it evolve in a unitary fashion to $t = t_e$,

$$\rho(t_e) = U^{-1}\,\rho(0)\,U = [a|\xi_1\rangle_{t_e} + b|\xi_2\rangle_{t_e}]\,[a^*\langle\xi_1|_{t_e} + b^*\langle\xi_2|_{t_e}] + \text{n.a.} \tag{20}$$

where U is usual time development operator for the interval $t = 0$ to $t = t_e$, and "n.a." stands for small terms coming from non-adiabaticity, which can induce hopping into the $|\xi_3\rangle$ and $|\xi_4\rangle$ states. Between $t = t_e$ and $t = t_e + t_h$ the time evolution is governed by the equation,

$$\frac{d}{dt}\rho(t) = -i[H(t_e), \rho(t)] - \Gamma[\zeta\,,\,[\zeta\,,\,\rho(t)]] = -i[H(t_e), \rho(t)] - \Gamma\,\{\rho(t) - \zeta\rho(t)\zeta\} \tag{21}$$

where the noise term is of the same form as given in Eq. (11), the constant Γ sets the level of the noise, and ζ is one of the error matrices, with $\zeta^2 = 1$. After evolution under Eq. (21) to time $t_e + t_h$ the density matrix is of the form,

$$\rho(t_e+t_h) = P_e(t_e+t_h)\,\exp(-iHt_h)\,U^{-1}\,\rho(0)\,U\,\exp(iHt_h)\,[1-\Gamma t_h]\,P_e(t_e+t_h) \\ +[1-P_e(t_e+t_h)]\,\rho_A\,[1-P_e(t_e+t_h)] + O(\Gamma^2) \qquad (22)$$

where the first term on the righthand side comes from the first part of the noise term in Eq. (21), which by itself would have produced a factor $\exp(-\Gamma t_h)$ for large Γt_h, and the other term, involving a matrix ρ_A that we shall not give explicitly, comes from the second part of the noise term that involves the ζ matrices.

Next we decode, by adiabatically changing the parameters back to their values at $t = 0$. This is a unitary process implemented by the time evolution operator $\overline{U}$ for the interval $t = t_e + t_h$ to $t = 2t_e + t_h$. We note that $\overline{U}P_e(t_e+t_h)\overline{U}^{-1} = P_0$, since the unitary evolution processes never cause a transition from one of the states $|\xi_j\rangle_t$ to another, and the relative phase factors from the time evolution cancel in the construction of the projection operators. Thus we obtain,

$$\rho(2t_e+t_h) = P_0\,\overline{U}\,\exp(-iHt_h)\,U^{-1}\,\rho(0)\,U\,\exp(iHt_h)\,\overline{U}^{-1}\,(1-\Gamma t_h)\,P_0 \\ +(1-P_0)\,\overline{U}\,\rho_A\,\overline{U}^{-1}\,(1-P_0) + O(\Gamma^2) \qquad (23)$$

In the $|\xi_1\rangle_0$, $|\xi_2\rangle_0$ subspace the unitary matrix $\overline{U}\,\exp(-iHt_h)\,U^{-1}$ is diagonal. Thus if we measure the second (control) variable and obtain unity (or no change from the initial value) there can have been no bitflip in the state of the first (information) state. Then, if we wish we can fine-tune the hang time, t_h, so that $\overline{U}\,\exp(-iHt_h)\,U^{-1}$ in the $|\xi_1\rangle_0$, $|\xi_2\rangle_0$ subspace is a phase factor times the unit matrix. In this case we have recaptured the initial state of the information bit perfectly, using implementable transformations that are independent of the information itself.

The above process can be characterized as being a limited error correction, in that we have corrected the errors that could have been created in the part of the final density matrix in which the value of the control bit is unity. In the case that the measurement of the control bit gives zero (this probability is itself of order Γt_h), then we can work out the explicit form for the second term on the righthand side of Eq. (23), and find a unitary operation that converts it to a multiple of the initial density matrix. This operation depends on which of the four error operators is used for the matrix ζ. Thus if we extended Eq. (21) to include several error terms at once, then, since we would not know which was operative in a given case, we could not implement the correction. Corrections of all of the error operators require encoding protocols involving more qubits, as discussed in Ref. [15], and in much subsequent literature.

However, if we had replaced the double-commutator on the righthand side of Eq. (21) by a sum of terms, one for each of our protectable error operators, the solution would still have the useful property of error detection. In this case we can view the result in Eq. (23) as preservation for the part where the control bit is unity, and with error detection in the cases in which measurement of the control bit gives zero. When we measure zero for the control bit, we must discard the information in the channel. But even a single redundant parallel channel with the same input data would reduce the probability of information loss to the level of order $(\Gamma t_h)^2$. Since the latter is the essential level of error for the case in which measurement of the control bit gives unity, this is the best result obtainable using our kind of mechanism.

In our numerical example, we simply solve Eq. (21) for the density matrix evolution in the presence of noise, beginning at time $t = 0$ and ending at time $t = 2t_e + t_h$, without breaking into the separate factors of the discussion above. This then includes every piece of Eq. (23), but with the additional corrections due to: a) Small non-adiabaticity (hopping) due to the fact that time taken by the encoding process is not infinite. b) Errors that occur from the noise interaction during encoding-decoding, due to the fact that the encoding is not instantaneous. c) The terms of higher order in Γt_h, which matter when the probability of more than one error during the time t_h becomes appreciable. For our example, we choose the initial state for the information bit to be simply $|1\rangle$. Figure 6 shows the probability of a bit flip error arising from an error operator $\zeta = \sigma_x \otimes \mathbb{1}$ plotted as a function of the average number of applications of the error, Γt_h, and on the same plot the results for the case in which the information is not encoded. We have also checked numerically to see that in cases in which we begin with a superposition of $|1\rangle$ and $|0\rangle$ for the information bit, and fine tune t_d to an appropriate value, the entire state, including the relative phase angle in the superposition, is preserved to the same degree as the bit-error values plotted in Fig. 6.

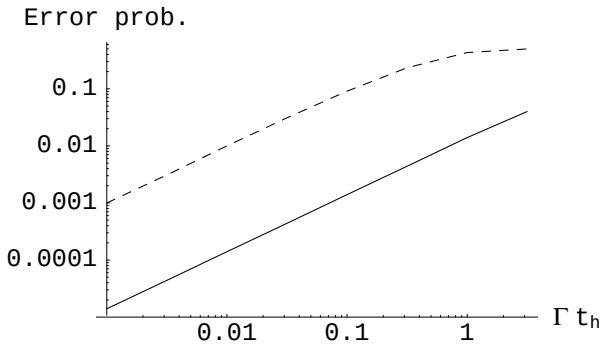

FIG. 6. The solid curve gives the bitflip error probability as a function of Γt_h for the case in which the encoding and decoding mechanisms are in place. The encoding time was taken to be $t_e = 0.01 \times t_h$. The dashed curve gives the error generated with the same noise source but without the encoding.

IV. CONCLUSION

We have considered two related topics in the study of closed and open quantum system evolution that go beyond one-body system Hamiltonians by including interaction terms between the system particles. In the first topic, we have shown that entangled states that are symmetric under the permutation of the basis states can be generated via adiabatic transitions from initially factorisable states which have this symmetry. In a two-qubit system, the outcome is the symmetrical Bell state $\frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)$, while the three-qubit analogue is the set of two W states. Noise in moderate quantitites generally reduces entanglement over time, and for such cases the limit of large noise sees entanglement remaining zero at all times. However, for the special case of "side-blind" noise that leads to motional narrowing, the creation of entanglement is preserved despite strong coupling to an environment.

In our second topic, we have shown that for small noise levels, we can use an entangling-pause-unentangling cycle to store and recover quantum data in a two qubit system, with error probability quadratic rather than linear in the parameter ΓT where Γ is the error rate for unencoded data and T is the time elapsed. Calculations of encoding that utilise the entanglement give a concrete demonstration of the compatibility of two requirements on the encoding time: a) that it be long enough achieve adiabaticity; b) that it be short enough to limit the effects of noise during the encoding process.

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